Name Nourhan Ibrahim , ID g00091476

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Exam II

Ayman Badawi

37/37 Excellent ++

QUESTION 1. (Show the work, 5 points)

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & -2 \end{bmatrix}$.

(i) If possible, find A^{-1}.

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -2 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \underset{2R_1+R_3 \to R_3}{\overset{R_1+R_2 \to R_2}{\sim}} \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{-R_2+R_1 \to R_1} \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array}\right] -R_3+R_1 \to R_1 \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array}\right]$$

by staring, $\underset{\sim}{R_2 \leftrightarrow R_3}$ $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -1 & -1 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$

$$A^{-1} = \begin{bmatrix} -2 & -1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(ii) **(Show the work, 3 points)** Find the solution set to $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$

$$\underbrace{A^{-1}A}_{I_3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

$$= 2 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$$

solution set = $\{(-4, 2, 4)\}$

QUESTION 2. **(Show the work, 6 points)** Given A is 4×4 such that

$$A \xrightarrow{R_1 \leftrightarrow R_3} B \xrightarrow{4R_3} C \xrightarrow{3R_1 + R_4 \to R_4} D = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & -1 & -1 & 8 \\ 0 & -3 & 0 & 7 \\ 0 & -2 & -2 & 1 \end{bmatrix}$$

$|B| = -|A|$

$|C| = -4|A|$

$|D| = -4|A|$

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 2 & -1 & -1 & 8 \\ 0 & -3 & 0 & 7 \\ 0 & -2 & -2 & 1 \end{vmatrix} \quad \begin{matrix} R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \\ 2R_1 + R_4 \to R_4 \end{matrix} \quad E = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 9 \\ 0 & 0 & 3 & 10 \\ 0 & 0 & 0 & 3 \end{vmatrix} \quad R_1 \leftrightarrow R_2 \; (-1) \quad F = \begin{vmatrix} 2 & 0 & 0 & 9 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 10 \\ 0 & 0 & 0 & 3 \end{vmatrix}$$

$|E| = -4|A|$ $\qquad$ $F = 4|A|$

(i) Find $|A|$ $\qquad$ $|F| = (2)(1)(3)(3) = 18$

$|F| = 4|A|$

$18 = 4|A|$ $\qquad$ $|A| = 18/4 = 9/2$

(ii) $|2C|$ $\qquad$ $|C| = -4|A| = -4(9/2) = -18$

$= 2^4 |C|$

$= 2^4(-18) = -288$

(iii) $|CD|$ $\qquad$ $|C| = |D| = -4|A| = -18$

$= |C| \cdot |D|$

$= (-18)(-18) = 324$

QUESTION 3. **(Show the work, 3 points)** Let $A = \begin{bmatrix} a & b & c \\ d & 2 & 4 \\ e & 1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3+a & b & c \\ d & 2 & 4 \\ e & 1 & 7 \end{bmatrix}$. If $|A| = 10$, what is $|B|$?

$|A| = (-1)^{1+1}(a)\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} + (-1)^{1+2}(b)\begin{vmatrix} d & 4 \\ e & 7 \end{vmatrix} + (-1)^{1+3}(c)\begin{vmatrix} d & 2 \\ e & 1 \end{vmatrix} = 10$

$|B| = (-1)^{1+1}(a+3)\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} + (-1)^{1+2}(b)\begin{vmatrix} d & 4 \\ e & 7 \end{vmatrix} + (-1)^{1+3}(c)\begin{vmatrix} d & 2 \\ e & 1 \end{vmatrix}$

$= 3\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} + \underbrace{a\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} + (-1)(b)\begin{vmatrix} d & 4 \\ e & 7 \end{vmatrix} + (c)\begin{vmatrix} d & 2 \\ e & 1 \end{vmatrix}}_{|A|} = 10$

$= 3\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} + 10$

$= 3((2 \times 7) - (4 \times 1)) + 10$

$= 3(10) + 10 = 30 + 10 = 40$ $\qquad$ $|B| = 40$

E1: $\left[\begin{array}{ccc|c} 0 & -2 & -2 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{-1/2 R_1} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$

$\xrightarrow{1/3 R_2} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{3R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$x_3 = 0 \quad \therefore x_2 = 0$

$x_1 \in \mathbb{R}$ free variable

or $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ if we do $(-R_2+R_1 \to R_1)$

For Question 5:

QUESTION 4. (Show the work, 6 points) The augmented matrix of a system of linear equations is

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 6 & 0 \\ -a & 1 & b & c & 4 \\ -c & 0 & -4c & -24 & 12 \end{array}\right]$$

$3\times4 \Rightarrow$ infinite

$aR_1+R_2\to R_2$
$cR_1+R_3\to R_3$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 6 & 0 \\ 0 & 1 & 4a+b & 6a+c & 4 \\ 0 & 0 & 0 & 6c-24 & 12 \end{array}\right]$$

a) For what values of a, b, c will the system have exactly one solution (one vector only)?

no such solution, 3×4 $4>3$ $m>n$, thus no one solution /unique
there is a free variable (x_3) (not all variables are leading)
at least one $\Rightarrow$ less equations than variables, (x_3 will be free)

b) For what values of a, b, c will the system have infinitely many solutions?

$6c-24(x_4)=12$ $[c\neq4]$ $a\in\mathbb{R}$
$6c-24\neq0$ $6c\neq24$ $b\in\mathbb{R}$
$c\in\mathbb{R}-\{4\}$

c) For what values of a, b, c will the system have no solutions?

$6c-24(x_4)=12$ $0=12$ (wrong) zero = non zero
$6c-24=0$ $\Rightarrow$ no solution $a\in\mathbb{R}$
$6c=24$ $c=4$ $b\in\mathbb{R}$
thus, $c=4$ / $c=\{4\}$

QUESTION 5. (Show the work, 6 points) Let $A=\begin{bmatrix}1&2&2\\0&4&0\\0&0&4\end{bmatrix}$ Find all eigenvalues of A and all eigenspaces of A.

$|\alpha I_3 - A| = 0$ already upper triangular

$$\left|\begin{bmatrix}\alpha&0&0\\0&\alpha&0\\0&0&\alpha\end{bmatrix}-\begin{bmatrix}1&2&2\\0&4&0\\0&0&4\end{bmatrix}\right| = \left|\begin{bmatrix}\alpha-1&-2&-2\\0&\alpha-4&0\\0&0&\alpha-4\end{bmatrix}\right|=0$$

$C_A(\alpha)=(\alpha-1)\overbrace{(\alpha-4)(\alpha-4)}^{(\alpha-4)^2}=0$
$\alpha=1$ (repeated once) $\alpha=4\Rightarrow$ (repeated twice)

E_1
Eigen spaces (In the back)

~~$E_1\Rightarrow\left[\begin{array}{ccc|c}0&-2&-2&0\\0&-3&0&0\\0&0&-3&0\end{array}\right]\xrightarrow{-\frac{1}{2}R_1}\left[\begin{array}{ccc|c}0&1&1&0\\0&-3&0&0\\0&0&-3&0\end{array}\right]\xrightarrow{3R_1+R_2\to R_2}\left[\begin{array}{ccc|c}0&1&1&0\\0&0&3&0\\0&3&0&0\end{array}\right]$~~

~~$\xrightarrow{\frac{1}{3}R_2}\left[\begin{array}{ccc|c}0&1&1&0\\0&0&1&0\\0&3&0&0\end{array}\right]\xrightarrow{-R_2+R_1\to R_1}\left[\begin{array}{ccc|c}0&1&0&0\\0&0&1&0\\0&3&0&0\end{array}\right]\xrightarrow{-3R_1+R_3\to R_3}\left[\begin{array}{ccc|c}0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]$~~

~~$x_2+x_3=0\Rightarrow x_2=-x_3$~~
~~$3x_3=0\Rightarrow x_3=0$~~
$3x_2=0$
$x_2=0$ $x_1\in\mathbb{R}$ $E_1=$ solution set $=\{(x_1,0,0)\mid x_1\in\mathbb{R}\}$
$=\{x_1(1,0,0)\mid x_1\in\mathbb{R}\}$
$=\text{span}\{(1,0,0)\}$

$E_4\Rightarrow\left[\begin{array}{ccc|c}3&-2&-2&0\\0&0&0&0\\0&0&0&0\end{array}\right]$ ← free ← free $\xrightarrow{\frac{1}{3}R_1}\left[\begin{array}{ccc|c}1&-2/3&-2/3&0\\0&0&0&0\\0&0&0&0\end{array}\right]$

$x_1-2/3x_2-2/3x_3=0$ $x_2\in\mathbb{R}$
$x_1=2/3x_2+2/3x_3$ $x_3\in\mathbb{R}$

$E_4=$ solution set $=\{(2/3x_2+2/3x_3, x_2, x_3)\mid x_2,x_3\in\mathbb{R}\}$
$=\{x_2(2/3,1,0)+x_3(2/3,0,1)\mid x_2,x_3\in\mathbb{R}\}$
$=\text{span}\{(2/3,1,0),(2/3,0,1)\}$

QUESTION 6. (i) Let $A = \begin{bmatrix} 3 & 13 \\ 1 & 4 \end{bmatrix}$. Then A^{-1} $\frac{1}{(3\times4)-(1\times13)}$ $\begin{bmatrix} 4 & -13 \\ -1 & 3 \end{bmatrix}$

$-1 \begin{pmatrix} 4 & -13 \\ -1 & 3 \end{pmatrix} = \begin{bmatrix} -4 & 13 \\ 1 & -3 \end{bmatrix}$

(a) $\begin{bmatrix} -4 & 13 \\ 1 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -13 \\ -1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -13 \\ -1 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & 13 \\ 1 & -4 \end{bmatrix}$

(ii) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. Then $|A+B| =$

$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{vmatrix}$

(a) 0 (b) 1 (c) 4 (d) 9

$(1)(3)(3)$

(iii) Let $T : R^2 \to R^2$ be a linear transformation such that $T(a_1, a_2) = (2a_1 + a_2,\ 8a_1 + 4a_2)$. Then the eigenvalues of A are

(a) 2, 4 (b) -2, -4 (c) 2, 3 (d) 0, 6

$\begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix}$

$(\alpha-2)(\alpha-4) - (-1)(-8)$

$(\alpha-2)(\alpha-4) - 8$

$\alpha^2 - 2\alpha - 4\alpha + \cancel{8} - 8$

$\alpha(\alpha-6)$ $\quad \alpha^2 - 6\alpha$

$\left| \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} \right|$

$\begin{vmatrix} \alpha-2 & -1 \\ -8 & \alpha-4 \end{vmatrix}$

(iv) Consider the following system $\begin{array}{ccc} 0 & 1 & 1 \end{array}$ $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_2 + a_3 \\ b_2 + b_3 \\ c_2 + c_3 \end{bmatrix}$, where $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$. Then

3x3 $\quad$ → ma $\quad$ a1 or b2 or c3

(a) The system has no solutions

(b) the system has infinitely many solutions and (0, 1, 1) is a solution.

(c) (0, 1, 1) is the only solution for the system.

(d) give us a break, the question makes no sense at all.

a b ~~c~~ d $\quad$ many or no solution ✓ not

determent = 0 c

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com

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